A Family of Equations Derived from the Moves to 2-Dimensional Foams (Intelligence of Low-dimensional Topology)

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A Family of Equations Derived from the Moves to 2-Dimensional Foams

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1 Introduction

This paper focus upon the eight moves to 2-foams that are derived by considering the decomposition of the 4-ball into a product of simplices. These decompositions, in turn, correspond to codimension-2 singularities of knotted trivalent graphs, and also the boundaries of chains in a homology theory that encompasses group and quandle homology. Here two series of equations are derived. The first series consists of four equations and includes the Yang-Baxter equation (YBE). The second series consists of eight equations and includes the Zamolochikov tetrahedral equation. Some solutions are proposed that correspond to computing the boundary maps of certain chains. This focus comes from the fourth section of my talk May 2017. I believe that slides are posted at the RIMS website. I also have posted an updated version on my webpage. Personal and financial acknowledgements appear at the end of this paper.

A fundamental principle in the formulations of categorifications is that even when two things are naturally isomorphic it is unnatural to consider them equal. Instead, we study their "sameness" by examining relationships among the isomorphisms that relate them. In the case of diagrams that represent the same knotting, say, of knotted trivalent graphs (KTGs) or handlebody knots (HBKs), we can formulate a finite set of moves that relate the equivalent diagrams, and understand these as local pictures of embedded 2-dimensional foams in 4-space. In general there is a theory of n-dimensional foams and codimension 2 embeddings thereof that is formulated in four steps.

First, a space Y^n is defined that is a model for the local structure of the foam. Second, a model for local crossings is formulated. A crossing will be understood as an isolated point that occurs at the intersection of strata when the foam is projected into codimension 1. Third, the crossings of $(n+1)$-dimensional foams are used to formulate an essential set of moves for n-foams. Fourth, these crossings are described as chains in a homology theory that encompasses group and rack homology.

In this note, I will not have space to describe the full set of Reidemeister/Roseman moves to 2-foams. However, I point out that the list given in [2] is incomplete. There are additional moves that involve the twist vertex which are not presented there. That error will not affect the current work which is more or less self-contained. Please also consult [1] for more information.

An organizational sketch follows. In Section 2, the axiomatics of the operations of multiplication and conjugation in a group are examined. Those axioms will be expressed diagrammatically, and the oriented moves to diagrams are interpreted as crossings of foams. In Section 3, the definition of the space Y^n is given, and in general, local pictures of crossings are defined. Importantly, the crossings of $(n+1)$-foams correspond to some of the moves for n-foams. The unification of group and quandle homology is outlined. The eight fundamental moves to 2-foams are presented. In Section 4, the original four moves to knotted trivalent graphs (KTG) are reinterpreted as abstract tensor diagrams as are the eight moves to foams. The foam moves are also depicted as 3-dimensional polytopes with pentagonal, hexagonal, and square faces. The square faces are sometimes commutators; other times they correspond to the associator. In Section 5, a simple solution to these systems is given via the homological considerations.

2 Algebraic Preliminaries

Let G denote a group and consider conjugation as a separate operation that is denoted $(\triangleleft)$. The group operation $(\cdot)$ will be denoted by juxtaposition when no ambiguity arises. Note also that we can restrict to a subset that is closed under multiplication and conjugation. We have the following four properties:

A $(ab)c = a(bc)$;

YI $(ab) \triangleleft c = (a \triangleleft c)(b \triangleleft c)$;

IY $(a \triangleleft b) \triangleleft c = a \triangleleft (bc)$;

III $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

These properties are schematized in Fig. 1 as moves to KTGs.

The moves are directed; the directions allow one to keep track of the orientations in homological boundaries. Thus, this author considers each of A, YI, IY, and III to have a well-defined left-hand and right-hand side. In fact, within these pictures boundaries are computed. The text will soon discuss boundaries. For the time being, consider the A-move as a chain (a,b,c); the YI-move is a chain $(a, b)|c$; the IY-move is a chain $a|(b, c)$, and the III-move is a chain $a|b|c$. boundaries of chains are computed using group homology and a graded Leibniz rule.

It is well-known that the A-move corresponds to a tetrahedron and the III-move corresponds to a cube when the planar pictures are dualized: the dual to a trivalent vertex is a triangle; the dual to a crossing is a square. Less apparent, but nonetheless easy to observe, is that each of YI and IY correspond to triangular prisms: $\Delta \times [0, 1]$ and $[0, 1] \times \Delta$ respectively. Figure 2 indicates the prismatic structure while simultaneously indicating each move as a broken surface diagram of a local crossing for a knotted 2-foam.

3 The space Y^n, local crossings and homology

Let $\Delta^{n+1} = \{\vec{x} \in \mathbb{R}^{n+2} : \sum x_i = 1 \ \& \ 0 \leq x_i\}$ denote the standard simplex. The space $Y^n \subset \Delta^{n+1}$ is defined as follows: $Y^0 = (\frac{1}{2}, \frac{1}{2})$. Take $\Delta^n_j =$

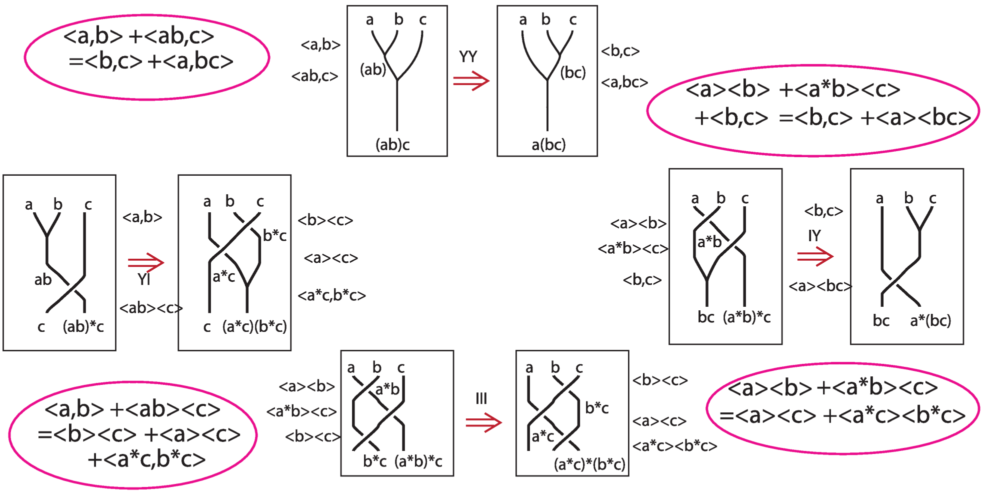


Figure 1: The four moves to KTGs that correspond to the boundaries of 3-chains

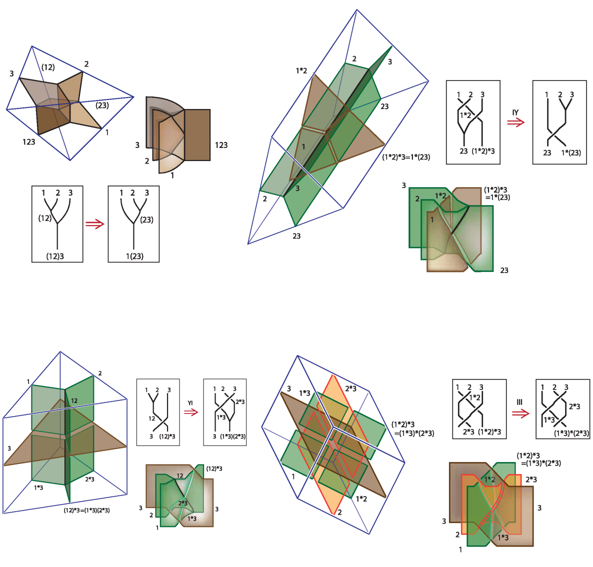


Figure 2: The local pictures for crossings of 2-foams fit inside prisms

$\{\vec{x} \in \Delta^{n+1} : x_j = 0\}$. Embed a copy, $Y_j^{n-1} \subset \Delta_j^n$. Cone $\cup_{j=1}^{n+2} Y_j^{n-1}$ to the barycenter $b = \frac{1}{n+2}(1, 1, \ldots, 1)$ of Δ^{n+1}. Define,

$$Y^n = C\left(\cup_{j=1}^{n+2} Y_j^{n-1}\right).$$

Then the space Y^1 is a neighborhood, Y, of a trivalent vertex. The space Y^2 is the local picture of the associativity relation when envisioned as a time-elapsed manifestation of a trivalent vertex moving past another. It is the carrier of the $6j$-symbol in the definition of Turaev-Viro invariants. It also is known as the IH-move.

An *n-dimensional foam (with boundary)* or simply *an n-foam* is a compact topological space X that locally modeled upon Y^n. Specifically, for every point $x \in X$, there is a neighborhood $N(x)$ and a homomorphism from $N(x)$ to a neighborhood of a point $y \in Y^n$. If y is non-singular, then x is a manifold-like point. In general, a point $y \in Y^n$ has a neighborhood that is homeomorphic to the product of a lower dimensional disk and a point in a lower dimensional Y^k. Of course, the boundary of an n-foam is an $(n-1)$-foam.

Since $Y^n \subset \Delta^{n+1}$, we can use this embedding to define a local crossing diagram.

Take

$$\left[\cup_{\ell=1}^{k}\left(\Delta^{j_1} \times \cdots \times Y^{j_\ell - 1} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{\ell\}\right],$$

and project this into $\mathbb{R}^{n+1}$. The factor ℓ is in the $(n+2)$nd coordinate and represents the relative height of each Y.

$$\left[\left(Y^{j_1-1} \times \Delta^{j_2} \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{1\}\right],$$

$$\left[\left(\Delta^{j_1} \times Y^{j_2-1} \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{2\}\right],$$

$$\cdots$$

$$\left[\left(\Delta^{j_1} \times \Delta^{j_2} \cdots \times Y^{j_\ell-1} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{\ell\}\right],$$

$$\cdots$$

$$\left[\left(\Delta^{j_1} \times \Delta^{j_2} \cdots \times \Delta^{j_\ell} \times \cdots \times Y^{j_k-1}\right) \subset \mathbb{R}^{n+1} \times \{k\}\right].$$

These project to a 0-dimensional multiple point in $\prod_{\ell=1}^{k} \Delta^{j_\ell} \subset \mathbb{R}^{n+1}$. Here we are interested in the local pictures for KTGs and for 2-foams.

Case 1: Local pictures for crossings of KTGs. There are only two ways to express a 2-disk as a product of simplices. It is either a triangle Δ or the product of intervals: a square $\Delta^1 \times \Delta^1$. In the latter case, the crossing picture is exactly the standard depiction of a knot crossing. In the former, the space Y lives as the dual to the triangle.

Case 2: Local crossings of 2-foams. There are four local pictures: The space Y^2 itself represents a "crossing" in that its vertex is 0-dimensional and isolated from other crossings. It is the junction of the six sheets and four seams of Y^2. Each seam has a neighborhood of the form $\mathsf{Y} \times \mathsf{I}$. In the case of the YI move, a transverse 2-dimensional disk crosses over the seam of $\mathsf{Y} \times \mathsf{I}$. For an IY move, the transverse sheet crosses under $\mathsf{Y} \times \mathsf{I}$. The III-move, when viewed from the current time-elapsed perspective, is the standard triple point figure that arrises by comparing the differing sides of the Reidemeister type III move. In the YI and IY cases, the crossing point has valence 5. It is the junction of the before and after seams of the $\mathsf{Y} \times \mathsf{I}$, and the three legs of Y crossing the transverse disk. The vertex of the III-move has valence 6 and coincides with the origin in the intersection of the three coordinate planes $\{(x, y, z) : xyz = 0\}$.

In general, we envision a grouping of an n-letter sequence as a decomposition of the n-ball into a product of simplices, or alternately, the local picture of a crossing of an $(n-1)$-dimensional foam. Meanwhile, the grouping can be interpreted as a family of binary trees in which a positive half-twist is inserted among the bottom branches of the trees. For a given grouping, Each binary tree can be "reassociated" using the A-move in, perhaps, many different ways. In addition, the crossings of the half-twists can commute with the Y-type vertices via IY or YI moves. And three successive crossings that form a triangle can be interchanged using the III move.

A given half-twisted family of trees has multiple planar diagrammatic representative. Each can be represented as a composition of words of the form $(\mathsf{II}\cdots\mathsf{IYI}\cdots\mathsf{I})$ or $(\mathsf{II}\cdots\mathsf{IXI}\cdots\mathsf{I})$. For each such composition, one can dualize the planar picture as a tiling by triangles (duals to Y) and squares (duals to X). Alternative representations are composed as moves across tetrahedra, triangular prisms, or cubes. Unions of these moves form 4-dimensional polytopes that are also prismatic. The pattern continues.

Let us digress on compositions of words and moves to KTGs for a moment. We can express the relations A, YI, IY, and III as follows.

A	(YI) $\circ$ (Y) $\Rightarrow$ (IY) $\circ$ (Y).
YI	(YI) $\circ$ (X) $\Rightarrow$ (IX) $\circ$ (XI) $\circ$ (YI)
IY	(XI) $\circ$ (IX) $\circ$ (YI) $\Rightarrow$ (IY) $\circ$ (X)
III	(XI) $\circ$ (IX) $\circ$ (IX) $\Rightarrow$ (IX) $\circ$ (IX) $\circ$ (XI)

We would be remiss in our duty to the reader if we did not point out that the III move in this form is a version of the Yang-Baxter equation.

We give a more algebraic description that depends upon partitioning. Let $g_1, g_2, \ldots, g_n$ denote elements of a group G. We will partition these into subsets $\langle g_1, \ldots, g_{\ell_1}\rangle$ $\langle g_{\ell_2+1}, \ldots, g_{\ell_1+\ell_2}\rangle$ $\ldots$ $\langle g_{[\sum_{j=1}^{k-1}\ell_j+1]}, \ldots, g_{[\sum_{j=1}^{k}\ell_j]}\rangle$. We consider such partitioned set of group elements to represent a generating n chain. Here, of course, $\sum_{j=1}^{k}\ell_j = n$.

To simplify the notation, we let i stand in for g_i. The partitioning then looks like $\langle 1, \ldots, \ell_1\rangle$ $\langle \ell_2+1, \ldots, \ell_1+\ell_2\rangle \ldots \left\langle \left[\sum_{j=1}^{k-1}\ell_j+1\right], \ldots, \left[\sum_{j=1}^{k}\ell_j\right]\right\rangle$.

Using a group-theoretic boundary operator, we define

$$\begin{aligned}
&\partial\langle j+1, j+2, \ldots, j+k\rangle \\
&\quad = \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \\
&\quad + \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle \\
&\quad + (-1)^k\langle j+1, \ldots, j+k-1\rangle.
\end{aligned}$$

In particular,

$$\partial\langle j+1\rangle = \triangleleft(j+1)\llcorner\!\lrcorner - \llcorner\!\lrcorner.$$

This is extended to a partitioned sequence by means of the Leibniz rule.

$$\partial(PQ) = (\partial P)Q + (-1)^{\dim P}P(\partial Q).$$

It is easy to show that $\partial \circ \partial = 0$ since conjugation and multiplication obey properties A, YI, IY, and III.

Since we are most interested in the relationships among 2-foams, we will compute the boundaries of $\langle a,b,c,d\rangle$ through $\langle a\rangle\langle b\rangle\langle c\rangle\langle d\rangle$

$$\begin{array}{rcl}
\partial\langle a,b,c,d\rangle & = & \langle b,c,d\rangle - \langle ab,c,d\rangle + \langle a,bc,d\rangle - \langle a,b,cd\rangle + \langle a,b,c\rangle; \\
\partial\langle a,b,c\rangle\langle d\rangle & = & \langle b,c\rangle\langle d\rangle - \langle ab,c\rangle\langle d\rangle + \langle a,bc\rangle\langle d\rangle - \langle a,b\rangle\langle d\rangle \\
 & & -\langle a \lhd d, b \lhd d, c \lhd d\rangle + \langle a,b,c\rangle; \\
\partial\langle a,b\rangle\langle c,d\rangle & = & \langle b\rangle\langle c,d\rangle - \langle ab\rangle\langle c,d\rangle + \langle a\rangle\langle c,d\rangle \\
 & & +\langle a \lhd c, b \lhd c\rangle\langle d\rangle - \langle a,b\rangle\langle cd\rangle + \langle a,b\rangle\langle c\rangle; \\
\partial\langle a,b\rangle\langle c\rangle\langle d\rangle & = & \langle b\rangle\langle c\rangle\langle d\rangle - \langle ab\rangle\langle c\rangle\langle d\rangle + \langle a\rangle\langle c\rangle\langle d\rangle \\
 & & +\langle a \lhd c, b \lhd c\rangle\langle d\rangle - \langle a,b\rangle\langle d\rangle \\
 & & -\langle a \lhd d, b \lhd d\rangle\langle c \lhd d\rangle + \langle a,b\rangle\langle c\rangle; \\
\partial\langle a\rangle\langle b,c,d\rangle & = & \langle b,c,d\rangle - \langle b,c,d\rangle \\
 & & -\langle a \lhd b\rangle\langle c,d\rangle + \langle a\rangle\langle bc,d\rangle - \langle a\rangle\langle b,cd\rangle + \langle a\rangle\langle b,c\rangle; \\
\partial\langle a\rangle\langle b,c\rangle\langle d\rangle & = & \langle b,c\rangle\langle d\rangle - \langle b,c\rangle\langle d\rangle \\
 & & -\langle a \lhd b\rangle\langle c\rangle\langle d\rangle + \langle a\rangle\langle bc\rangle\langle d\rangle - \langle a\rangle\langle b\rangle\langle d\rangle \\
 & & -\langle a \lhd d\rangle\langle b \lhd d, c \lhd d\rangle + \langle a\rangle\langle b,c\rangle; \\
\partial\langle a\rangle\langle b\rangle\langle c,d\rangle & = & \langle b\rangle\langle c,d\rangle - \langle b\rangle\langle c,d\rangle - \langle a \lhd b\rangle\langle c,d\rangle + \langle a\rangle\langle c,d\rangle \\
 & & +\langle a \lhd c\rangle\langle b \lhd c\rangle\langle d\rangle - \langle a\rangle\langle b\rangle\langle cd\rangle + \langle a\rangle\langle b\rangle\langle c\rangle; \\
\partial\langle a\rangle\langle b\rangle\langle c\rangle\langle d\rangle & = & \langle b\rangle\langle c\rangle\langle d\rangle - \langle b\rangle\langle c\rangle\langle d\rangle - \langle a \lhd b\rangle\langle c\rangle\langle d\rangle + \langle a\rangle\langle c\rangle\langle d\rangle \\
 & & +\langle a \lhd c\rangle\langle b \lhd c\rangle\langle d\rangle - \langle a\rangle\langle b\rangle\langle d\rangle \\
 & & -\langle a \lhd d\rangle\langle b \lhd d\rangle\langle c \lhd d\rangle + \langle a\rangle\langle b\rangle\langle c\rangle.
\end{array}$$

4 Moves to foams as abstract tensor diagrams

The local crossings of 3-foams are represented here as moves to 2-foams.

There are eight local pictures. We will envision these by examining the boundaries of each of the local pictures. Imagine that the letters (a,b,c,d) adorn the thin strings of the diagrams on the right hand side of each cross section. Follow the strings towards the left. Each Y junction represents a

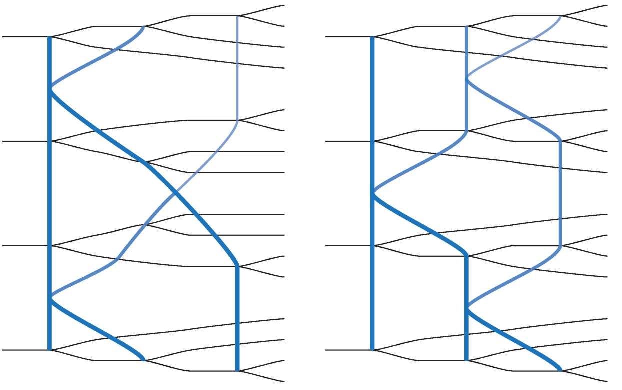


Figure 3: The YYY move that corresponds to Δ^4

binary multiplication. Each X junction can be expressed as a transposition, or more precisely as a braiding. The vertical steps on each figure are of the form A, YI, IY, III, or possibly a commutation of distant moves. The local pictures are given in the Figures: 3, 4, 5, 6, 7, 8, 9, and 10. These correspond respectively to groupings (a, b, c, d), $(a, b, c)|d$, $(a, b)|(c, d)$, $(a, b)|c|d$, $a|(b, c, d)$, $a|(b, c)|d$, $a|b|(c, d)$, and $a|b|c|d$.

5 Formulating and solving tensorial equations

The same figures will be illustrated without the movies of the 2-foams as the horizontal cross-sections. Before that, let us make an observation. The horizontal cross-sections are compositions of words in I, X, and Y. They are connected vertically by moves that correspond to the moves A, YI, IY, and III. In addition, there are commutators that occur because distant expressions of the Xs or Ys commute. Each of the atomic pieces A, YI, IY and III will be expressed as an abstract tensor. Respectively, the tensor operators are K, M, W or Ж. As such $V \otimes V \xleftarrow{\mathsf{K}} V \otimes V$, $V \otimes V \xleftarrow{\mathsf{M}} V \otimes V \otimes V$, $V \otimes V \otimes V \xleftarrow{\mathsf{W}} V \otimes V$, and $V \otimes V \otimes V \xleftarrow{\text{Ж}} V \otimes V \otimes V$. The last tensor operator Ж is a handmade version of the cyrillic letter that is pronounced "je." The vector space V is the direct sum of two vector spaces T and

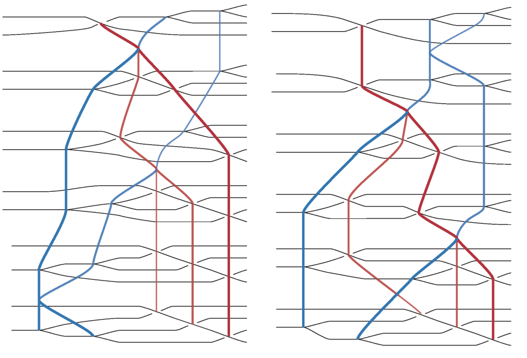


Figure 4: The YYI move that corresponds to $\Delta^3 \times \Delta^1$

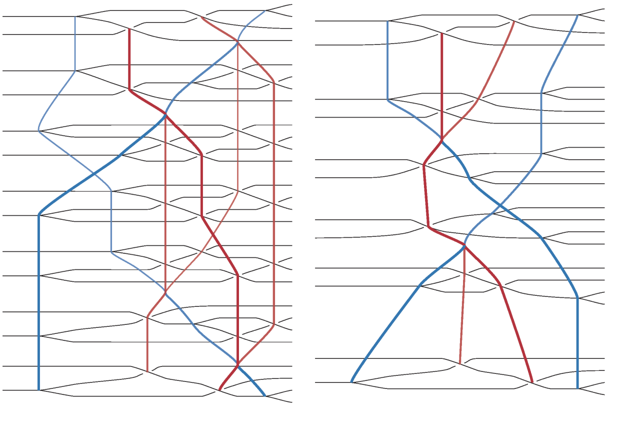


Figure 5: The YY foam move that corresponds to $\Delta^2 \times \Delta^2$

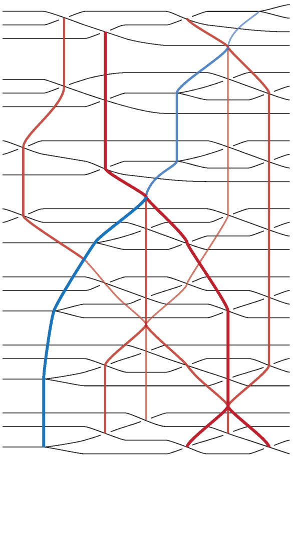


Figure 6: The YII foam move that corresponds to $\Delta^2 \times \Delta^1 \times \Delta^1$

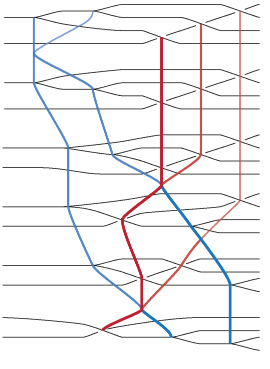


Figure 7: The IYY foam move that corresponds to $\Delta^1 \times \Delta^3$

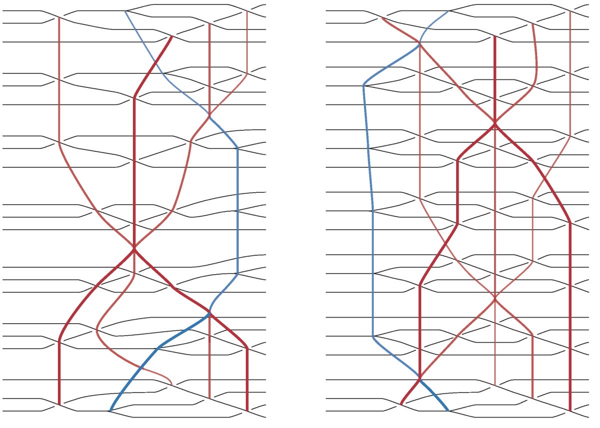


Figure 8: The IYI foam move that corresponds to $\Delta^1 \times \Delta^2 \times \Delta^1$

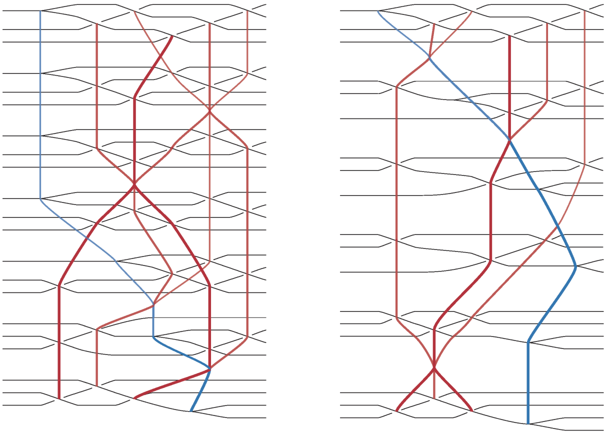


Figure 9: The IIY foam move that corresponds to $\Delta^1 \times \Delta^1 \times \Delta^2$

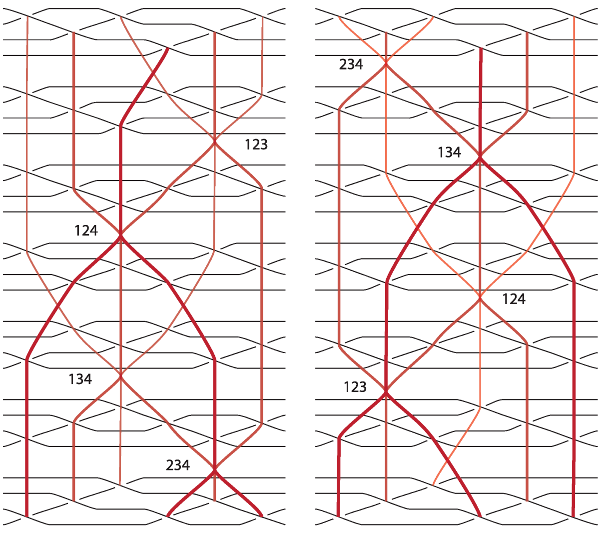


Figure 10: The IIII foam move that corresponds to $\Delta^1 \times \Delta^1 \times \Delta^1 \times \Delta^1$

S, called *together* and *separated*. The summands T and S each are free over pairs of elements of the group G.[1] In T we write the basis elements (a, b) and in S we write $a|b$. More generally, we may assume that T and S are free modules generated by pairs of elements (a, b) and $a|b$ respectively. Please note, there is no intension to confuse the notation $a|b$ with the bar resolution. They are different.

The figures 11 through 14 indicate the junctions at the moves, and they rewrite the relations induced from the foam moves as relationships among the tensor operators K, M, W, and Ӿ. In particular, we imagine the crossings (which are of the form X $\times [0, 1]$) and the multiplication operators (which are of the form Y $\times [0, 1]$) as the carriers of the basis of the tensor equations. The strings of the form X $\times [0, 1]$ correspond to bases of S, and those of the form Y $\times [0, 1]$ correspond to the bases in T.

These tensor equations are the family in which we are most interested. However, the moves A, YI, IY, and III also can be formulated as abstract

[1]More generally, we can assume that we have a free module over any algebraic structure for which the conditions A through III hold.

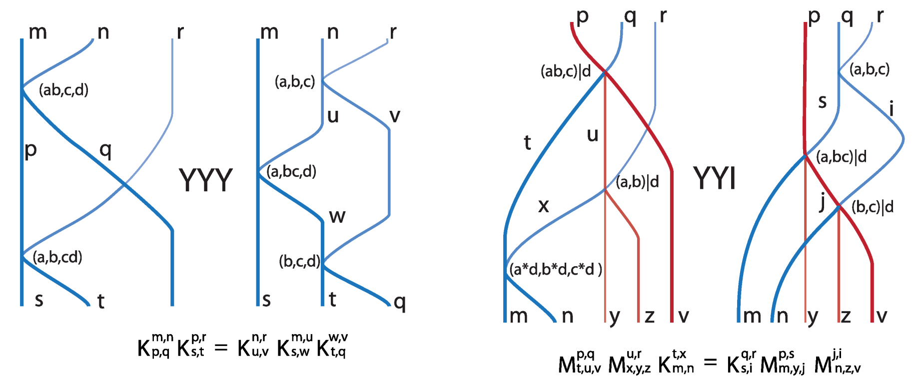


Figure 11: Abstract tensor equations YYY and YYI

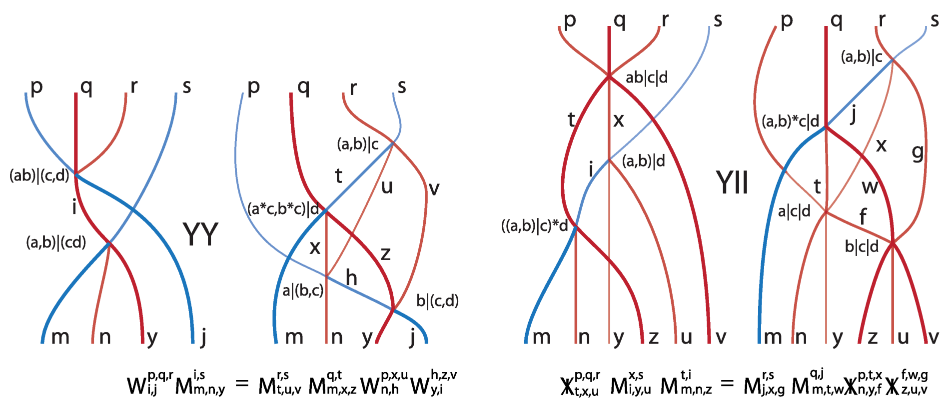
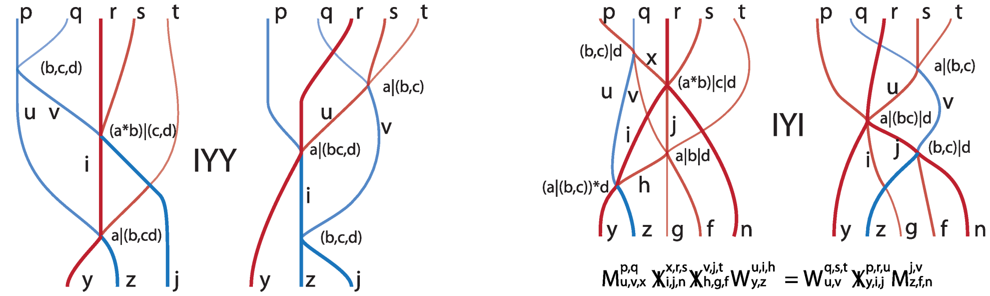


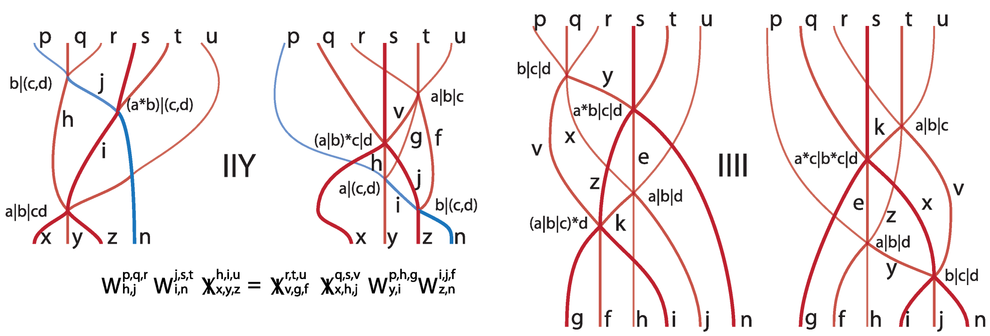
Figure 12: Abstract tensor equations YY and YII



$$K^{p,q}_{u,v} W^{v,r,s}_{i,j} \; W^{u,i,t}_{y,z} = W^{q,s,t}_{u,v} W^{p,r,u}_{y,i} K^{i,v}_{z,j}$$

$$M^{p,q}_{u,v,x} \text{Ж}^{x,r,s}_{i,j,n} \text{Ж}^{v,j,t}_{h,g,f} W^{u,i,h}_{y,z} = W^{q,s,t}_{u,v} \text{Ж}^{p,r,u}_{y,i,j} M^{j,v}_{z,f,n}$$

Figure 13: Abstract tensor equations IYY and IYI



$$W^{p,q,r}_{h,j} \; W^{j,s,t}_{i,n} \; \text{Ж}^{h,i,u}_{x,y,z} = \text{Ж}^{r,t,u}_{v,g,f} \; \text{Ж}^{q,s,v}_{x,h,j} \; W^{p,h,g}_{y,i} W^{i,j,f}_{z,n}$$

$$\text{Ж}^{p,q,r}_{v,x,y} \; \text{Ж}^{y,s,t}_{z,e,n} \; \text{Ж}^{x,e,u}_{k,h,j} \; \text{Ж}^{v,z,k}_{g,f,i} = \text{Ж}^{r,t,u}_{k,z,v} \; \text{Ж}^{q,s,k}_{g,e,x} \; \text{Ж}^{p,e,z}_{f,h,y} \; \text{Ж}^{y,x,v}_{i,j,n}$$

Figure 14: Abstract tensor equations IIY and IIII

tensors. For both A and III these formulations are no surprise. In the case of A the equations is the relation among the structure constants in an associative algebra. In the case of III, the equation is the Yang-Baxter relation written in tensorial form. These four equations appear in Table 1. Here and below, Einstein summation conventions hold: sums are taken

A	$\mathsf{Y}_{j}^{\ell,n}\mathsf{Y}_{i}^{j,k} = \mathsf{Y}_{p}^{n,k}\mathsf{Y}_{i}^{\ell,p}$
YI	$\mathsf{Y}_{d}^{a,b}\mathsf{X}_{e,f}^{d,c} = \mathsf{X}_{c,h}^{b,c}\mathsf{X}_{e,g}^{a,c}\mathsf{Y}_{f}^{g,h}.$
IY	$\mathsf{X}_{d,g}^{a,b}\mathsf{X}_{h,f}^{g,c}\mathsf{Y}_{e}^{d,h} = \mathsf{Y}_{i}^{b,c}\mathsf{X}_{e,f}^{a,i}.$
III	$\mathsf{X}_{d,e}^{a,b}\mathsf{X}_{i,h}^{e,c}\mathsf{X}_{f,g}^{d,i} = \mathsf{X}_{j,k}^{b,c}\mathsf{X}_{f,e}^{a,j}\mathsf{X}_{g,h}^{e,k}$

Table 1: The equations induced by the KTG moves

over repeated indices that appear as subscripts and superscripts.

We have the tensor equations for the 2-foam moves in Table 2.

YYY	$\mathsf{K}_{p,q}^{m,n}\mathsf{K}_{s,t}^{p,r} = \mathsf{K}_{u,v}^{n,r}\mathsf{K}_{s,w}^{m,u}\mathsf{K}_{t,q}^{w,v}$
YYI	$\mathsf{M}_{t,u,v}^{p,q}\mathsf{M}_{x,y,z}^{u,r}\mathsf{K}_{m,n}^{t,x} = \mathsf{K}_{s,i}^{q,r}\mathsf{M}_{m,y,j}^{p,s}\mathsf{M}_{n,z,v}^{j,i}$
YY	$\mathsf{W}_{i,j}^{p,q,r}\mathsf{M}_{m,n,y}^{i,s} = \mathsf{M}_{t,u,v}^{r,s}\mathsf{M}_{m,x,z}^{q,t}\mathsf{W}_{n,h}^{p,x,u}\mathsf{W}_{y,i}^{h,z,v}$
YII	$\boldsymbol{\mathsf{X}}_{t,x,v}^{p,q,r}\mathsf{M}_{y,u,v}^{x,s}\mathsf{M}_{m,n,z}^{t,i} = \mathsf{M}_{j,x,g}^{r,s}\mathsf{M}_{m,t,x}^{q,j}\boldsymbol{\mathsf{X}}_{n,y,f}^{p,t,x}\boldsymbol{\mathsf{X}}_{z,u,v}^{f,w,g}$
IYY	$\mathsf{K}_{u,v}^{p,q}\mathsf{W}_{i,j}^{v,r,s}\mathsf{W}_{y,z}^{u,i,t} = \mathsf{W}_{u,v}^{q,s,t}\mathsf{W}_{y,i}^{p,r,u}\mathsf{K}_{z,j}^{i,v}$
IYI	$\mathsf{M}_{u,v,x}^{p,q}\boldsymbol{\mathsf{X}}_{i,j,n}^{x,r,s}\mathsf{X}_{h,g,f}^{v,j,t}\mathsf{W}_{y,z}^{u,i,h} = \mathsf{W}_{u,v}^{q,s,t}\boldsymbol{\mathsf{X}}_{y,i,j}^{p,r,u}\mathsf{M}_{z,f,n}^{j,v}$
IIY	$\mathsf{W}_{h,j}^{p,q,r}\mathsf{W}_{i,n}^{j,s,t}\boldsymbol{\mathsf{X}}_{x,y,z}^{h,i,u} = \boldsymbol{\mathsf{X}}_{v,g,f}^{r,t,u}\boldsymbol{\mathsf{X}}_{x,h,j}^{q,s,v}\mathsf{W}_{y,i}^{p,h,g}\mathsf{W}_{z,n}^{i,j,f}$
IIII	$\boldsymbol{\mathsf{X}}_{v,x,y}^{p,q,r}\boldsymbol{\mathsf{X}}_{z,e,n}^{y,s,t}\boldsymbol{\mathsf{X}}_{k,h,j}^{x,e,u}\boldsymbol{\mathsf{X}}_{g,f,i}^{v,z,k} = \boldsymbol{\mathsf{X}}_{k,z,v}^{r,t,u}\boldsymbol{\mathsf{X}}_{g,e,x}^{q,s,k}\boldsymbol{\mathsf{X}}_{f,h,y}^{p,e,z}\boldsymbol{\mathsf{X}}_{i,j,n}^{y,x,v}$

Table 2: The equations induced by the 2-foam moves

Now we turn to solutions. For the series of equations in the Table 1, we formulate solutions as follows. Let V denote group algebra, or more generally a free module with basis elements of a group, or set with operations $(\cdot)$ and $(\triangleleft)$ that satisfy our main properties. Let $V \otimes V \xleftarrow{\mathsf{Y}} V$, and $V \otimes V \xleftarrow{\mathsf{X}} V \otimes V$ denote tensor operators of the form

$$\mathsf{Y}_{c}^{a,b} = \begin{cases} 1 & \text{if } c = ab, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{X}^{a,b}_{c,d} = \begin{cases} 1 & \text{if } c = b \ \& \ d = b^{-1}ab, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that these form a solution, but in fact, we will give a more sophisticated reason for this in a moment.

Theorem 1. *The tensor entries defined by*

$$\mathsf{K}^{p,q}_{r,s} = \begin{cases} 1 & \text{if } \exists\ a,b,c \text{ s.t. } \begin{cases} p = ((ab,c),0|0), \quad q = ((a,b),0|0), \\ r = ((a,bc),0|0), \& \ s = ((b,c),0|0). \end{cases} \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{M}^{i,j}_{p,q,r} = \begin{cases} 1 & \text{if } \exists\ a,b,c \text{ s.t. } \begin{cases} i = ((0,0),ab|c), \quad j = ((a,b),0|0), \\ p = ((a,b) \triangleleft c, 0|0), \quad q = ((0,0),a|c), \\ \& \ r = ((0,0),b|c), \end{cases} \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{W}^{i,j,\ell}_{p,q} = \begin{cases} 1 & \text{if } \exists\ a,b,c \text{ s.t. } \begin{cases} i = ((b,c),0|0), \quad j = ((0,0),(a \triangleleft b)|c), \\ \ell = ((0,0),a|b), \\ p = ((0,0),a|(bc)), \quad \& \ q = ((b,c),0|0), \end{cases} \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{X}\!\!\!\mathsf{X}^{p,q,r}_{s,t,u} = \begin{cases} 1 & \text{if } \exists\ a,b,c \text{ s.t. } \begin{cases} p = ((0,0),b|c), \quad q = ((0,0),(a \triangleleft b)|c), \\ r = ((0,0),a|b), \\ s = ((0,0),(a|b) \triangleleft c), \quad t = ((0,0),a|c), \\ \& \ u = ((0,0),b|c), \end{cases} \\ 0 & \text{otherwise;} \end{cases}$$

where, e.g., $a \triangleleft c = c^{-1}ac$, $(a,b) \triangleleft c = (a \triangleleft c, b \triangleleft c)$, *and* $(a|b) \triangleleft c = (a \triangleleft c)|(b \triangleleft c)$ *solve the system of equations in Table 2.*

Moreover, the tensor operators $\mathsf{Y}^{a,b}_{c}$ *and* $\mathsf{X}^{a,b}_{c,d}$ *that are defined above satisfy the equations contained in Table 1.*

Proof. We start with the conceptually easier case for the solutions X and Y. The tensor equations in Table 1 arise as a result of taking the formal boundaries of chains (a,b,c) (condition A), $(a,b)|c$ (condition YI), $a|(b,c)$ (condition IY), and $a|b|c$ (condition III). The subscripts and the superscripts in the expressions for $\mathsf{Y}^{a,b}$ and $\mathsf{X}^{a,b}_{c,d}$ correspond formally to taking the boundaries of the chains (a,b) and $a|b$. Thus we can see that the tensor operator in the expressions are, in fact, cocycles that are coboundaries. Essentially, we are taking the boundary twice, and thus the equalities hold.

More specifically in this case, the vector space (or module V in this case should be an expression as chains in $S_1 \oplus T_1$ where each summand is just a free module generated by the underlying group. In this case, the sub-, and superscripts should be listed as pairs of basis elements in the group.

Such a listing is given explicitly for the sub- and superscripts of K, M, W, and Ж. The equations of Table 2 are formulated by means of taking a formal boundary. The sub- and superscripts are the result of taking the boundary once more with the signs of the top and bottom being opposite. Thus these equations hold since the operators are, in fact, cocycles that are coboundaries. This completes the proof. □

In full generality, the boundaries of crossings of n-dimensional foams give 0-dimensional multiple points in the boundaries of prismatic sets. They are joined by arcs in this boundary. We can use these to formulate tensor equations that simultaneously include the higher dimensional versions of the Zamolodchikov equation and equations that formulate the higher dimensional Pachner moves. Meanwhile, using the underlying algebraic structure we obtain solutions precisely because a homology theory is formed.

Finally, we remark that equation YYY is a version of the Elliott-Biedenharn identity, and equation IIII is the Zamolochikov equation given in tensorial form. In [3], a similar collection of generalizations are given. But only extreme cases seem to coincide.

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